6.4 - Special Functions

nu

Bessel's equation of order v: $x^2y'' + xy' + (x^2 - v^2)y = 0$

Example: Find the general solution of the given differential equation on $(0, \infty)$.

$16x^2y'' + 16xy' + (16x^2 - 1)y = 0$

$$x^2y'' + xy' + \left(x^2 - \frac{1}{16}\right)y = 0$$

$$v^2 = 1/16 \Rightarrow v = \pm 1/4$$

solution: $\boxed{y = c_1 J_{1/4}(x) + c_2 J_{-1/4}(x)}$

Bessel Functions of the First and Second Kinds

Assuming a solution of the form $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ and substituting into Bessel's equation, we have

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r} + \sum_{n=0}^{\infty}(n+r)c_nx^{n+r} + \sum_{n=0}^{\infty}c_nx^{n+r+2} - \sum_{n=0}^{\infty}v^2c_nx^{n+r} = 0$$

$$\implies \sum_{n=0}^{\infty}[(n+r)(n+r) - v^2]c_nx^{n+r} + \sum_{n=0}^{\infty}c_nx^{n+r+2} = 0$$

Reindexing yields $\sum_{k=0}^{\infty}[(k+r)^2 - v^2]c_kx^{k+r} + \sum_{k=2}^{\infty}c_{k-2}x^{k+r} = 0$

For $k = 0$ we find $r = \pm v$ and $c_1 = 0$.

Then $c_k = -\dfrac{c_{k-2}}{(k+r)^2 - v^2}$, $k = 2, 3, 4, \ldots$

Note: We'll only have nonzero terms if k is even.

Relabel Let $2n = k$ with $r = \nu, -\nu$

$$c_{2n} = -\frac{1}{(2n+\nu)^2 - \nu^2} c_{2n-2}, \quad n = 1, 2, 3, \dots$$

$$4n^2 + 4n\nu + \cancel{\nu^2} - \cancel{\nu^2}$$

$$c_{2n} = -\frac{1}{2^2 n(n+\nu)} c_{2n-2}, \quad n = 1, 2, 3, \dots$$

$$n=1 \quad c_2 = -\frac{1}{2^2(1+\nu)} c_0$$

$$n=2 \quad c_4 = -\frac{1}{2^2 \cdot 2(2+\nu)} c_2 = \frac{1}{2^4 \cdot 2(2+\nu)(1+\nu)} c_0$$

$$n=3 : \quad c_6 = -\frac{1}{2^6 \cdot 2 \cdot 3(3+\nu)(2+\nu)(1+\nu)} c_0$$

In general, $c_{2n} = \dfrac{(-1)^n}{2^{2n} n! (1+\nu)(2+\nu) + \cdots + (n+\nu)} c_0$

A standard choice for c_0 is $c_0 = \dfrac{1}{2^\nu \Gamma(1+\nu)}$, where $\Gamma(1+\nu)$ is the gamma function.

Definition: The **gamma function** is the integral-defined function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

Note that $\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,dt$

$u = t^x \qquad dv = e^{-t}\,dt$

$du = x t^{x-1}\,dt \qquad v = -e^{-t}$

$$\Gamma(x+1) = \overset{0}{\cancel{-t^x e^{-t}\Big|_0^\infty}} + x\int_0^\infty t^{x-1} e^{-t}\,dt$$

$\Gamma(x+1) = x\,\Gamma(x)$ (we're using ν)

so $\Gamma(\underline{\nu}+1) = \underline{\nu}\,\Gamma(\underline{\nu})$

$\Gamma(\nu+2) = \Gamma(\underline{\nu+1}+1) = (\underline{\nu+1})\,\Gamma(\underline{\nu+1})$

$\Gamma(\nu+3) = \Gamma(\nu+2+1) = (\nu+2)(\nu+1)\,\Gamma(\nu+1)$

$\vdots$

$\Gamma(\nu+n+1) = (\nu+n)\cdots(\nu+2)(\nu+1)\,\Gamma(\nu+1)$

(Like a Generalized factorial)

so

$$c_{2n} = \frac{(-1)^n}{2^{2n+\nu}\, n!\,\Gamma(1+\nu+n)}, \qquad n = 1, 2, 3, \dots$$

(using a standard choice for c_0)

The function $y = x^{\nu} \sum_{n=0}^{\infty} c_{2n} x^{2n}$ is then

$$y = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} = J_{\nu}(x)$$

We also have that $J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1-\nu+n)} \left(\frac{x}{2}\right)^{2n-\nu}$

Definition: The functions $J_{\nu}(x)$ and $J_{-\nu}(x)$ defined above are **Bessel functions of the first kind** of order ν and $-\nu$, respectively and converge at least on $(0, \infty)$.

If ν is not an integer, then the general solution to Bessel's equation is $y = c_1 J_{\nu}(x) + c_2 J_{-\nu}(x)$.

Definition: The function $Y_{\nu}(x) = \dfrac{\cos(\nu\pi)J_{\nu}(x) - J_{-\nu}(x)}{\sin \nu\pi}$ (ν a non-integer) is a **Bessel function of the second kind**. It can be shown to be linearly independent with $J_{\nu}(x)$.

If ν is an integer, then it can be shown that $J_{\nu}(x)$ and $J_{-\nu}(x)$ are linearly dependent. However, $J_{\nu}(x)$ and $Y_{\nu}(x)$ are independent. As such, for an integer value of ν, we use $J_{\nu}(x)$ and $Y_{\nu}(x)$ (in this case, $Y_{\nu}(x)$ is actually defined by a limiting process as $\nu \to m$, where m is an integer). In this case we can give the general solution to Bessel's equation as $y = c_1 J_{\nu}(x) + c_2 Y_{\nu}(x)$.

Example: Find the general solution of the given differential equation on $(0, \infty)$.

$x^2 y'' + xy' + (x^2 - 1)y = 0$

$\longrightarrow \nu = \pm 1$ (integers)

$$\boxed{y = c_1 J_1(x) + c_2 Y_1(x}$$

Alternate forms of Bessel's equation:

Using a substitution we can find that the general solution to the DE $x^2y'' + xy' + (\alpha^2x^2 - \nu^2)y = 0$ is $y = c_1J_\nu(\alpha x) + c_2Y_\nu(\alpha x)$.

Definition: The preceding differential equation is called the **parametric Bessel equation of order ν**.

Definition: The differential equation $x^2y'' + xy' - (x^2 + \nu^2)y = 0$ is the **modified Bessel equation of order ν**. It can be converted to a Bessel equation by the substitution $t = ix$, where $i^2 = -1$; the result is...

Definition: The **modified Bessel function of the first kind** of order ν, $I_\nu(x) = i^{-\nu}J_\nu(ix)$.

The general solution to the modified Bessel equation of order ν (ν not an integer) is $y = c_1I_\nu(x) + c_2I_{-\nu}(x)$.

Definition: As before, for non-integer values of ν we define the **modified Bessel function of the second kind**, $K_\nu(x) = \dfrac{\pi}{2}\dfrac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi}$ and extend this definition to integer values using a limit process similar to that referenced when defining $Y_\nu(x)$.

And as before, the general solution to the modified Bessel equation of order ν (for ν an integer) is $y = c_1I_\nu(x) + c_2K_\nu(x)$. A parametric form of the modified Bessel equation and associated solution exists such as we saw above.

Example: Find the general solution of the given differential equation on $(0, \infty)$.

$x^2y'' + xy' - (2x^2 + 64)y = 0$

$\alpha = \sqrt{2}\ , \ \nu = \pm 8i$

$y = c_1 I_8(\sqrt{2}x) + c_2 K_8(\sqrt{2}x)$

non-integer ℝ ν:

$c_1 J_\nu(x) + c_2 J_{-\nu}(x)$

Integer ℝ ν:

$c_1 J_\nu(x) + c_2 Y_\nu(x)$

non-integer Im ν:

$c_1 I_\nu(x) + c_2 I_{-\nu}(x)$

integer Im ν:

$c_1 I_\nu(x) + c_2 K_\nu(x)$

Example: The general solution to

$$y'' + \boxed{\frac{1-2a}{x}}y' + \boxed{\left(b^2c^2x^{2c-2} + \frac{a^2 - p^2c^2}{x^2}\right)}y = 0\ , p \geq 0$$

is $y = x^a\left[c_1 J_p\left(bx^c\right) + c_2 Y_p\left(bx^c\right)\right]$. Use this to find the general solution of the given differential equation on $(0, \infty)$.

$xy'' + 3y' + xy = 0$

$$y'' + \boxed{\frac{3}{x}}\,y' + \boxed{}\,y = 0$$

$3 = 1 - 2a \Rightarrow a = -1$

$1 = b^2c^2x^{\underbrace{2c-2}} + \dfrac{1-p^2c^2}{x^2}$

$b^2c^2 = 1 \Rightarrow b^2 = 1$

(use $b = 1$)

$2c - 2 = 0$

$c = 1$

$\Rightarrow p = 1$

$p^2c^2 = 1$

(no $\dfrac{\text{stuff}}{x^2}$ on LHS)

$$\boxed{y = x^{-1}\left[c_1 J_1(x) + c_2 Y_1(x)\right]}$$

Legendre polynomials:

Definition: The differential equation

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0$$

is **Legendre's equation of order n**.

Note that $x = 0$ is an ordinary point. Starting with $y = \sum_{k=0}^{\infty} c_k x^k$,...

leads to $2c_2 + 6c_3x - 2c_1x + n(n+1)c_0 + n(n+1)c_1x$
$+ \sum_{j=2}^{\infty}\left\{(j+2)(j+1)c_{j+2} + \left[-j(j-1) - 2j + n(n+1)\right]c_j\right\}x^j = 0$

We find that $c_{j+2} = -\dfrac{(n-j)(n+j+1)}{(j+2)(j+1)}c_j,\ j = 2, 3, 4, \ldots$

$$c_2 = -\frac{n(n+1)}{2}c_0$$

$$c_4 = -\frac{(n-2)(n+3)}{4\cdot 3}c_2$$

This terminates if n is even.

$$\begin{aligned} c_3 &= -\frac{(n-1)(n+2)}{6}c_1 \\ c_5 &= -\frac{(n+4)(n-3)}{5 \cdot 4}c_3 \\ &= \frac{(n-3)(n-1)(n+2)(n+4)}{5!}c_1 \end{aligned}$$

This terminates if n is odd.

Traditional values for c_i are

For $n = 0$, $c_0 = 1$; for $n = 2,\ 4,\ 6,\ \ldots$, $c_0 = (-1)^{n/2}\dfrac{1 \cdot 3 \cdots (n-1)}{2 \cdot 4 \cdots n}$

For $n = 1$, $c_1 = 1$; for $n = 3,\ 5,\ 7,\ \ldots$, $c_1 = (-1)^{(n-1)/2}\dfrac{1 \cdot 3 \cdots n}{2 \cdot 4 \cdots (n-1)}$

Definition: For each value of n the result is a **Legendre polynomial of order *n***.

The first six Legendre polynomials are

$$P_0(x) = 1 \qquad P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \qquad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$